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Solitary waves in the $\phi^4+\lambda\phi^3$ model with and without dissipation

J Geicke

Centro Técnico Aeroespacial, Instituto de Estudos Avançados, 12200 São José dos Campos, São Paulo, Brazil

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Abstract. It is shown that in the $(1+1)$-dimensional real $\phi^4+\lambda\phi^3$ model a new non-oscillating kink-antikink solution exists. For $\lambda=0$ it reduces to a single ϕ^4 kink as the second constituent disappears at infinity. We calculate the repulsive force between kink and antikink, generated by the $\lambda\phi^3$ term. From this and the doublet state we find a quantitative approximation for the attractive force between a ϕ^4 kink and antikink. In the presence of a dissipative force $\alpha\phi_t$ the doublet solution is stationary while single-kink solutions with α- and λ-dependent velocity exist.

1. Introduction

Besides the well known ϕ^4 theory, an extension to the real $\phi^4+\lambda\phi^3$ model has also been considered in physics [1-4]. In particular, it has been used in the construction of the $(3+1)$-dimensional Friedberg-Lee model [1-3] in elementary particle physics. For this reason, and also because kink solutions do not exist for $\lambda\neq 0$, it is, already in $1+1$ dimensions, interesting to find and discuss its solitary-wave solutions.

The Lagrangian density of the model is

$$\mathscr{L}=\tfrac{1}{2}\phi_t^2-\tfrac{1}{2}\phi_x^2-V(\phi) \tag{1a}$$

where

$$V(\phi)=\tfrac{1}{4}\phi^4+\tfrac{1}{3}\lambda\phi^3-\tfrac{1}{2}\phi^2+V_0 \qquad \lambda=\text{real} \qquad V_0=\text{constant} \tag{1b}$$

is the potential energy density. To keep the notation simple we choose the coefficient of the self-coupling term $\frac{1}{4}\phi^4$ as $g^2=1$ and that of the mass term as $m^2=1$. This is possible without loss of generality since the field ϕ and space and time coordinates x and t can be rescaled†. When travelling wave solutions $\phi(x,t)=y(x-vt)$ are considered, the field equation reduces to the ordinary differential equation

$$\mathrm{d}^2y/\mathrm{d}z^2=y^3+\lambda y^2-y \qquad z=\gamma(x-vt) \qquad \gamma=(1-v^2)^{-1/2} \qquad v=\text{constant}. \tag{2}$$

† For positive g^2 and m^2: $\phi\to m\phi/g$, $x\to x/m$, $t\to t/m$ (and $\lambda\to gm\lambda$). If the mass term is positive [1, 3], say $\hat V(\hat\phi)=\frac{1}{4}\hat\phi^4+\frac{1}{3}\hat\lambda\hat\phi^3+\frac{1}{2}\hat\phi^2$ and $\hat\lambda^2>4$, then $\hat\phi=\phi+c$, where c solves $c^2+\hat\lambda c+1=0$, transforms $\hat V(\hat\phi)$ into $V(\phi)=\frac{1}{4}\phi^4+(\frac{1}{3}\hat\lambda+c)\phi^3-\frac{1}{2}(1-c^2)\phi^2+\frac{1}{4}c^4+\frac{1}{3}\hat\lambda c^3+\frac{1}{2}c^2$. Because of $\hat\lambda^2>4$, both solutions c are real and one is in the interval $-1<c<1$ such that $-\frac{1}{2}(1-c^2)<0$. $\hat\lambda^2>4$ is also the condition for $\hat V(\hat\phi)$ to have its extremum values at three different real values of $\hat\phi$.

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A characteristic difference between the $\phi^4+\lambda\phi^3$ model and pure ϕ^4 theory is that the term $\frac{1}{3}\lambda\phi^3$ destroys the symmetry under $\phi\to-\phi$. Therefore the two vacua of the theory (minima of V) in

$$\phi_\pm=-\tfrac{1}{2}\lambda\pm(1+\tfrac{1}{4}\lambda^2)^{1/2} \tag{3}$$

which are degenerate for $\lambda=0$, have different energies for $\lambda\neq0$. The perturbation $\frac{1}{3}\lambda\phi^3$ removes the degeneracy of the vacua and for this reason does not permit kink solutions to (2).

In § 2 we derive particular solutions to (2) and among them discuss a new kink-antikink doublet state and its dissociation for very weak coupling. The doublet solution is possible because the term λy^2 in (2) generates a repulsive force, which at a certain distance compensates the attractive force between kink and antikink. Both forces are calculated in § 3. In § 4 a frictional term $-\alpha\phi_t=\alpha\gamma v\,\mathrm{d}y/\mathrm{d}z$ is introduced on the LHS of (2). Then single-kink solutions, with α- and λ-dependent velocity, can be re-established in analogy with damped and driven ϕ^4 kinks.

2. An exact kink–antikink solution

A first integral to (2) is

$$\tfrac{1}{2}(\mathrm{d}y/\mathrm{d}z)^2=\tfrac{1}{4}y^4+\tfrac{1}{3}\lambda y^3-\tfrac{1}{2}y^2+\tfrac{1}{4}C=:P(y) \tag{4}$$

which can be integrated in terms of elliptic functions. For certain values of the integration constant $C=C(\lambda)$, such that two of the zeros of $P(y)=0$ are equal, particular integrals of (2) exist. The conditions for two equal roots $y_1=y_2=a$ of $P(y)=0$ are

$$a(a^2+\lambda a-1)=0 \tag{5a}$$

$$C=3a^4+\tfrac{8}{3}\lambda a^3-2a^2. \tag{5b}$$

Equation (5a) has the solutions

$$a=0\qquad a=-\tfrac{1}{2}\lambda\pm(1+\tfrac{1}{4}\lambda^2)^{1/2}=:y_\pm. \tag{6}$$

Using in (5b) the values (6) for a, one gets from (4)

$$\int\mathrm{d}w/[w(w^2+Ew+F)^{1/2}]=\pm\int\mathrm{d}z/\sqrt{2} \tag{7a}$$

$$w=y-a \tag{7b}$$

$$E=4a+\tfrac{4}{3}\lambda \tag{7c}$$

$$F=6a^2+4\lambda a-2. \tag{7d}$$

From the denominator of (7a) one sees that (at least for real λ) $P(y)=0$ cannot have three (or four) equal roots. This would require $F=0$ (and $E=0$), in contradiction to $F=-2$ for $a=0$ and $F=2a^2+2>2$ for $a=y_\pm$. There is only one other special case possible, namely that $P(y)=0$ has two double roots. Then it is necessary that $E^2=4F$, i.e. $a^2+\frac{2}{3}\lambda a-1-\frac{2}{9}\lambda^2=0$. The latter equation is compatible with (5a) only for $\lambda=0$ and $a=y_\pm=\pm1$. One obtains the well known ϕ^4 kink.

For $a=0$ the solution

$$y=w=-4/[(8+\tfrac{16}{9}\lambda^2)^{1/2}\sin(z-z_0)-\tfrac{4}{3}\lambda] \tag{8}$$

is oscillating. Further, for real z it is either not real (if the integration constant z_0 is not real) or it has poles (if z_0 is real), and therefore will not be discussed further.

For $a=y_\pm$ (7a, d) yield $F=2a^2+2>0$ and

$$w=-g/[1-(g-E)^2/4F] \qquad g=\exp[s(\tfrac{1}{2}F)^{1/2}(z-z_0)] \qquad s=\pm 1. \tag{9}$$

This solution was recently mentioned by Su and Gu [4]. They refer to it as 'another specific soliton solution' which 'cannot reduce to the kink-antikink solution when $\lambda\to 0$', and consider it only in the half-interval where the argument in the exponential is positive. This restriction suppresses an important part of the solution (9). We will now discuss the solitary-wave content of this solution.

For $a=y_\pm$ (7c) yields

$$E=E(\lambda,y_\pm)=E_\pm(\lambda)=-\tfrac{2}{3}\lambda\pm 2(4+\lambda^2)^{1/2} \tag{10a}$$

that is $E_+(\lambda)>0$ and $E_-(\lambda)<0$ for all real λ. In order that the solution (9) be real and finite for all real z, the exponential g must be real and of the same sign as $-E(\lambda)$, for real z. Therefore, the (so far) arbitrary constant z_0 must be real if $a=y_-$ ($E=E_-<0$), while it must have an imaginary part Im $z_0=\mathrm{i}\pi(2/F)^{1/2}$ if $a=y_+$ ($E=E_+>0$). As (2) is translationally invariant, we set Re $z_0=0$ and thus

$$g=g_+=-\exp(s(\tfrac{1}{2}F)^{1/2}z) \qquad \text{if } \lambda>0 \tag{10b}$$

$$g=g_-=+\exp(s(\tfrac{1}{2}F)^{1/2}z) \qquad \text{if } \lambda<0. \tag{10c}$$

Secondly, the condition $E_\pm^2>4F$ must be fulfilled, yielding

$$\tfrac{4}{9}\lambda^2\mp\tfrac{8}{3}\lambda(4+\lambda^2)^{1/2}>\mp 4\lambda(4+\lambda^2)^{1/2}. \tag{10d}$$

For $\lambda>0$ the inequality (10d) is true with the negative sign ($a=y_+$), and for $\lambda<0$ with the positive sign ($a=y_-$). In conclusion, (9) provides the solitary-wave solution

$$y=w(E_+,g_+)+y_+ \qquad \text{for } \lambda>0 \tag{11a}$$

$$y=w(E_-,g_-)+y_- \qquad \text{for } \lambda<0. \tag{11b}$$

We now suppose $\lambda\geqslant 0$ and discuss (11a). In the limit $\lambda=+0$ we find $E_+=4$, $F=4$, $g_+=-\exp(s\sqrt{2}z)$ and

$$y(z)=1-16g_+/(-g_+^2+8g_+)=\tanh(sz/\sqrt{2}-\ln\sqrt{8}). \tag{12}$$

The solution (11a) reduces to a ϕ^4 kink ($s=1$) at $x(t)=vt+\sqrt{2}\gamma^{-1}\ln\sqrt{8}$ or to an antikink ($s=-1$) at $x(t)=vt-\sqrt{2}\gamma^{-1}\ln\sqrt{8}$. For $0<\lambda\ll 1$ (11a) is, up to first order in λ,

$$y\approx 1-\tfrac{1}{2}\lambda+16(1-\tfrac{1}{2}\lambda)/[g_+-8+\tfrac{4}{3}\lambda+8\lambda/(3g_+)]$$

$$g_+\approx-\exp(s\sqrt{2}(1-\tfrac{1}{4}\lambda)z). \tag{13a}$$

For $g_+\approx -8$, i.e. at

$$x=x_1(t)\approx vt+s\ln 8/[\sqrt{2}\gamma(1-\tfrac{1}{4}\lambda)] \tag{13b}$$

one finds the centre of an approximate kink ($s=1$) or antikink ($s=-1$), in agreement with (12). On the other hand, when $8\lambda/3g_+\approx -8$, i.e. at

$$x=x_2(t)\approx vt+s\ln(\tfrac{1}{3}\lambda)/[\sqrt{2}\gamma(1-\tfrac{1}{4}\lambda)] \tag{13c}$$

we have the centre of an approximate antikink ($s=1$) or kink ($s=-1$). For both choices, $s=1$ and $s=-1$, the kink and antikink move with equal velocity v, $-1<v<1$, at a constant distance

$$d(\lambda, v)=|x_2-x_1|\approx \ln(24/\lambda)/[\sqrt{2}\gamma(1-\tfrac{1}{4}\lambda)]. \tag{14}$$

To conclude, the solution ($11a$) represents a doublet formed from a kink and an antikink. The kink is always to the right of the antikink. This is because the solution ($11a$) is also 'attracted' by the other vacuum y_- and satisfies $y(z)\leqslant y(\pm\infty)=y_+$. Finally, one should mention the factor $1/\gamma$ in formula (14) which describes the Lorentz contraction of the doublet state.

Analogous results are obtained from ($11b$) for $-1\ll\lambda<0$. But kink and antikink are interchanged and $\ln\lambda\to\ln(-\lambda)$, as (2) is invariant under $y\to -y$ and $\lambda\to-\lambda$. That the kink is now to the left of the antikink is compatible with, and necessary because of, $y(z)\geqslant y(\pm\infty)=y_-$.

The doublet state (11) is qualitatively different from the oscillating bound states, exact and with amplitude between 0 and 2π in sine-Gordon (SG) theory ('breather') [5], and approximate or asymptotic in ϕ^4 theory [6-9]. There kink and antikink periodically exhibit their non-linear interactions. The solution (11) is also different from the SG kink-antikink doublet [5] which describes a kink and antikink approaching each other with opposite velocities, colliding at some finite time and afterwards again emerging with their original velocity and shape. One concludes that the term λy^2 in (2) is responsible for a repulsive force F_r between a kink and an antikink. When both are at distance (14) from each other, F_r compensates the attractive force F_a commonly supposed to exist between them. For $\lambda=0$, kink and antikink cannot exist at a finite constant distance from each other, as there is only the attractive force between them. Therefore the constituent at the position ($13c$) disappears at $x=\pm\infty$ as $s\ln|\lambda|$.

One can understand the limit $\lambda=+0$, i.e. the different asymptotic values of ($11a$) and (12) also, by considering the motion of a point mass in a potential $V(y)=-\frac{1}{4}y^4-\frac{1}{3}\lambda y^3+\frac{1}{2}y^2$. The equation of motion is $\mathrm{d}^2y/\mathrm{d}t^2=-\mathrm{d}V/\mathrm{d}y$. When the point mass has total energy corresponding to the particular solution ($11a$) and a velocity $\mathrm{d}y/\mathrm{d}t<0$ at time $t=t_0$, it reaches the turning point $y_{\min}>y_-$ at a (for $\lambda>0$ finite) time $t_1>t_0$, and then returns to $y=y_+$, where it arrives asymptotically at $t=\infty$. For $\lambda\to 0$ one gets $y_{\min}\to y_-$ and $t_1\to\infty$; i.e. for $\lambda=0$ the point mass tends asymptotically to y_- and does not return to $y=y_+$. A corresponding analogy exists for $\lambda\leqslant 0$.

3. Quantitative expressions for the forces

That the term λy^2 in (2) simulates a repulsive force between the kink and antikink in the solution (11)—independently of the sign of λ—can be seen from the transformation

$$y=u-\tfrac{1}{3}\lambda. \tag{15a}$$

Then (2) becomes

$$\mathrm{d}^2u/\mathrm{d}z^2=u^3-(1+\tfrac{1}{3}\lambda^2)u+\tfrac{1}{3}\lambda+\tfrac{2}{27}\lambda^3. \tag{15b}$$

We consider $-1\ll\lambda\ll 1$ and neglect terms of second and third order in λ. To order λ, kink and antikink do not experience any x-dependent deformation by the force $\frac{1}{3}\lambda$ [10]. In (11) they satisfy boundary conditions at infinity which correspond to $u=\pm1-\frac{1}{6}\lambda$ and are compatible with the force $\frac{1}{3}\lambda$. (The extremum value of the solution

(11), of course, is different from $y_{\pm}$ already as $\tanh(d/2\surd 2)\approx 1-(\frac{1}{6}|\lambda|)^{1/2}$, $d=(1/\surd 2)\ln(24/|\lambda|)$.) Thus the only effect of the force $\frac{1}{3}\lambda$ in order λ is an acceleration [10]; for a kink it has the same sign as λ. For the antikink the sign is opposite because $y\to -y$ in (2), or $u\to -u$ in (15*b*), is equivalent to $\lambda\to-\lambda$. Since in (11), for $\lambda>0$, the kink is to the right of the antikink, and for $\lambda<0$ to the left, the force $\frac{1}{3}\lambda$ has a repulsive effect. Obviously, it is independent of the distance between kink and antikink, and does not lead to a relative force between two kinks, or between two antikinks.

The repulsive force acting on the RHS constituent in the solution (11) can be written in the form

$$F_r=\tfrac{1}{3}|\lambda|. \tag{16a}$$

Let $q\gg 1$ denote the general distance between kink and antikink, and $F_a(q)$ the attractive force acting on the RHS constituent in its rest system. For $q=d(\lambda, v=0)$ one finds from (14) and (16*a*)

$$F_a(q=d)=F_a[(1/\surd 2)\ln(24/|\lambda|)]=-F_r=-\tfrac{1}{3}|\lambda|. \tag{16b}$$

Using (14) with $1-\frac{1}{4}\lambda\approx 1$ and expressing λ by q one gets

$$F_a(q)=-8\exp(-\sqrt{2}q). \tag{16c}$$

This result is valid for $-1\ll\lambda\ll 1$ or $\ln|\lambda|\ll -1$, and thus for sufficiently large q such that the single kinks can be distinguished in the doublet state. The LHS constituent in the doublet experiences the forces $-F_r$ and $-F_a(q)$ (where q again is considered positive).

Equations (16*a*) and (16*c*) show that the resulting force $F_r+F_a(q)$ between kink and antikink is attractive when their distance q is less than the equilibrium distance d, and repulsive when q is greater than d. Thus the solution (11) is unstable against small changes of the distance d and cannot be considered as a bound state.

In (13*a*) one can also observe a slight increase of the kink width by the factor $G=1/(1-\frac{1}{4}\lambda)$ in order λ and for $q=d(\lambda, v)$. An interpretation of this is that those parts of kink and antikink which have a slightly smaller distance are attracted more strongly than those with a larger distance than d. For general $q\gg 1$ the factor G has the form

$$G\approx 1/[1-6\exp(-\sqrt{2}q)]\approx 1+6\exp(-\sqrt{2}q). \tag{16d}$$

To complete our results we show that the driven ϕ^4 equation

$$\mathrm{d}^2\hat{y}/\mathrm{d}z^2=\hat{y}^3-\hat{y}+\tfrac{1}{3}\lambda \tag{17a}$$

for $-(4/27)^{1/2}<\frac{1}{3}\lambda<(4/27)^{1/2}$ has an exact particular solution which is similar to the solution (9) of (2). A first integral is

$$(\mathrm{d}\hat{y}/\mathrm{d}z)^2=\tfrac{1}{2}\hat{y}^4-\hat{y}^2+\tfrac{2}{3}\lambda\hat{y}+\hat{C}=:Q(\hat{y}). \tag{17b}$$

For the choice $\hat{C}=3b^4/2-b^2$ of the integration constant the equation $Q(\hat{y})=0$ has a double root and (17*a*) the particular solution

$$\hat{y}(z)=b-\hat{g}/[1-(\hat{g}-4b)^2/4\hat{F}]$$
$$\hat{g}=\exp(\pm(\tfrac{1}{2}\hat{F})^{1/2}(z-z_0)) \qquad \hat{F}=6b^2-2. \tag{17c}$$

The constant $b=b(\lambda)$ solves the cubic equation $b^3-b+\frac{1}{3}\lambda=0$, and the solutions $b=b_{\pm}=\pm 1-\frac{1}{6}\lambda+\mathrm{O}(\lambda^2)$ may be chosen.

The solution (17c) can be evaluated in an analogous way to equation (9). For $0<\lambda \ll 1$, $b=b_+$, $z_0=\mathrm{i}\pi(2/\hat{F})^{1/2}$ one finds a result which differs from (13a) to order λ only by the additive constant $\frac{1}{3}\lambda$. Therefore the results (16c) and (16d) can also be obtained from the solution (17c). The important (although expected) conclusion to be drawn is the following. Since the exact solutions (9) and (17c) satisfy different boundary conditions $y(-\infty)=y(\infty)\approx \pm 1-\frac{1}{2}\lambda$ and $\hat{y}(-\infty)=\hat{y}(\infty)\approx \pm 1-\frac{1}{6}\lambda$, but otherwise correspond with each other to order λ, they indicate that a constant x-independent shift $\phi(x,t)\to\phi(x,t)+\text{constant}$ of the kink and antikink does not influence their interaction in leading order.

4. 'Damped kinks'

Finally, we mention that (2) with a frictional term, i.e.

$$\mathrm{d}^2y/\mathrm{d}z^2+\alpha\gamma v\,\mathrm{d}y/\mathrm{d}z=y^3+\lambda y^2-y \qquad \alpha>0 \tag{18a}$$

has kink solutions of the form

$$y=a+h/[1+\exp(f(z-z_0))] \tag{18b}$$

with α- and λ-dependent velocity and real integration constant z_0. This is suggested by corresponding results for the damped and driven ϕ^4 equation [11] and the transformation (15a). The ansatz (18b) yields a kink or its antikink for

$$\begin{aligned} &a=y_- \qquad h=y_+-y_-=(\lambda^2+4)^{1/2} \\ &f=\pm(2+\tfrac{1}{2}\lambda^2)^{1/2} \qquad v=\mp\lambda/\sqrt{2} \end{aligned} \tag{19a}$$

and two kink solutions or their antikinks for

$$\begin{aligned} &a=0 \qquad h=y_- \text{ or } h=y_+ \\ &2f^2=1-\lambda h>0 \qquad \alpha\gamma v=(1+f^2)/f. \end{aligned} \tag{19b}$$

All these solutions correspond closely to the damped and driven ϕ^4 kinks [11, 12] and therefore need no further discussion. They exist because the frictional loss of energy is equal to the decrease of the kinks' potential energy as previously explained for ϕ^4 theory with friction [13]. This compensation mechanism determines the absolute value and the sign of the velocity. (The solutions (18b) and (19) exist also for $\alpha<0$. But this situation is less physical since the term $\alpha\gamma v\,\mathrm{d}y/\mathrm{d}z$ now adds energy to the system. Each kink has the opposite velocity to that for $\alpha>0$ and, when moving, increases its potential energy.)

Clearly, (18a) cannot have solutions with arbitrary velocity, since the frictional term destroys the relativistic invariance. The doublet (11) solves (18a) only for $v=0$.

5. Summary

In the $\phi^4+\lambda\phi^3$ model a new kink-antikink doublet solution exists, where both constituents move at a constant distance from one another. It is unstable against small changes of the equilibrium distance, i.e. such perturbations are expected to cause complete dissociation or approximation of the two constituents. The serious instability is that for distances $q>d$. Namely, the asymptotic values of the solution (11), y_+ for $\lambda>0$ and y_- for $\lambda<0$, do not correspond to the vacuum (global minimum) of the theory, but only to the second (local) minimum of $V(\phi)$. Therefore the solution (11) cannot be related to the non-topological soliton field [1-3] which has been studied as a model for quark confinement and which will exist only in the presence of quarks or other suitable fields.

The physical content of (11) is that it permits an interesting insight into the interaction between a ϕ^4 kink and its antikink, leading to the exponential (approximative) law (16*c*) for the attractive force $F_a(q)$ and to the enhancement factor (16*d*) for the width. For this result it is relevant that both constituents in the doublet (11) in order λ, $-1\ll\lambda\ll 1$, do not experience any x-dependent deformation by the repulsive force (16*a*) and that an x-independent shift $\phi(x,t)\to\phi(x,t)+\text{constant}$, equal for kink and antikink, does not change their interaction.

Another complementary result is that the ϕ^4 kink and its antikink, destroyed by the perturbation $\lambda\phi^3$, can be re-established in the form (18*b*) and (19*a*) by a frictional term. This may be of practical interest for problems where the ϕ^4 model has been applied and where relativistic invariance is not essential, e.g. for non-linear excitations in linear polymeric chains such as polyacetylene [14]. It is probably realistic to permit small ϕ^3 perturbations of the ϕ^4 potential and an energy transfer from the kinks to the chain (by friction). Also, two other kinks and their antikinks (18*b*) and (19*b*), connecting a minimum of V in $y=y_-$ or in $y=y_+$ with the maximum in $y=0$ exist. All the solutions are possible because friction provides a simple mechanism to 'absorb' the potential energy, which becomes 'free' during the motion of a kink if this connects non-degenerate extrema of the potential.

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